On the moduli space of generalized holomorphic maps

Stefano Chiantese

Freie Universität Berlin,
FB Mathematik und Informatik,
Arnimallee 3, 14195 Berlin, Germany,
Email: chiantes@mi.fu-berlin.de.

ABSTRACT

We compute the anomalies of the topological A and B models with target space geometry of Hitchin's generalized type. The dimension of the moduli space of generalized holomorphic maps is also computed, which turns out to be equal to the total anomaly if the moduli space is unobstructed. We obtain this result by identifying the infinitesimal deformations of such maps and by using the Grothendieck-Riemann-Roch formula.

1 Introduction

The aim of this paper is to make some concrete preliminary steps towards an understanding of the moduli space of holomorphic maps from curves to a generalized Calabi-Yau manifold[1]. Via the localization principle, this has been shown by Kapustin [1] to be the relevant moduli space of the topological A and B models with Hitchin's generalized complex geometry [2]. Such maps generalize the notion of holomorphic embeddings of curves into a Calabi-Yau manifold. These give the moduli space over which correlators of the topological A model on a Calabi-Yau manifold localize. In this case it is well known that coupling the A model to worldsheet gravity is of fundamental importance. In fact, there are no genus $g > 1$ correlators for a fixed worldsheet geometry. This can be seen by selection rules which come from the anomalies in the twisted theory. From a mathematical point of view this is due to the fact that there are no holomorphic maps from a genus $g > 1$ Riemann surface to a Calabi-Yau manifold.

These considerations tell us that for computing correlators of topological string theory with generalized complex geometry, one has to tackle first the problem of the anomalies in the twisted theory. In this paper we compute the vector and axial anomalies of the A and B models on a generalized Kähler manifold [3], which has to satisfy a geometry constraint so that the twist can be performed [4]. The key facts one has to take into account is that left and right fermions are defined in terms of two different complex structures of the target space geometry, and the Dirac operator in the kinetic terms of the untwisted theory becomes the Dolbeault operator as a result of the twist. As a consequence, the anomalies are given in terms of the indices of the Dolbeault operator coupled to the holomorphic tangent spaces of the two complex structures.

We also compute the dimension of the moduli space of holomorphic maps from curves to a generalized Kähler manifold. To this end one has to identify the deformations of such maps. We find that the deformations (obstructions) lie in the zero (first) Čech cohomology of sections of the maximal isotropic subbundle of a generalized complex structure. If the moduli space is unobstructed, its virtual or expected dimension

[1]The reason why we mainly focus on generalized Calabi-Yau manifolds will become clear later on in the paper. However, for the topological A and B models the generalized Calabi-Yau condition can be properly relaxed as explained in the next paragraph.

coincides with the total anomaly when the above geometry constraint is taken into account. Moreover, we find that if there are no marked points on the Riemann surface, generalized Calabi-Yau threefolds assume a special role. In fact, the virtual dimension becomes zero regardless the genus of the Riemann surface. In the final part of the paper we use this result to speculate on a possible generalization of Gromow-Witten theory.

2 The anomalies of the twisted theories

We want to compute the anomalies of the A and B models when the target space is a generalized Kähler manifold. To this end, it is useful to recall the case of the A model with its usual target space geometry which is Kähler.[2] In such a case, while we do not need to impose an anomaly cancellation condition to define the model, the presence of an axial anomaly in the twisted σ-model leads to a selection rule for the correlation functions.[3] The computation of this anomaly is closely related to that of the axial anomaly of the untwisted σ-model, where it is given by the index of the Dirac operator. The main difference is that the Dirac operator becomes the Dolbeault operator as a result of the twist. The anomaly is then given by the index of the Dolbeault operator, which is related to the dimension of the moduli space of holomorphic maps via the Grothendieck-Riemann-Roch formula (GRR).

The main fact one has to take into account to extend these considerations to topological models with Hitchin's generalized geometries is that left and right moving fermions are defined in terms of two different complex structures of the target space. As a result, one is forced to cancel a vector anomaly to perform the A twist [4], while for the usual Kähler geometry the vector symmetry is never anomalous. The considerations of the precedent paragraph then suggest us that the topological models should have axial and vector anomalies.

[2]For simplicity, we do not consider the Landau-Ginzburg model. In this case, the σ-model can be twisted assigning zero vector charge to all chiral superfields.

[3]There is also a selection rule coming from the vector symmetry. Here, we are principally interested in the axial anomaly since it is related to the dimension of the moduli space of holomorphic maps. The situation will change for generalized geometries as we will see below.

To compute these anomalies we need to know the kinetic terms of the fields and their R-charges. The anomaly is given by a mismatch of the fermion zero modes, which is expressed in terms of the indeces of the operators appearing in the kinetic terms. These indices are then computed by the index theorem. The R-charges are given by the following table

$$\begin{array}{c|cc} & q_{\mathrm{V}} & q_{\mathrm{A}} \\ \hline \mathcal{P}_+\psi_+ & -1 & -1 \\ \overline{\mathcal{P}}_+\psi_+ & +1 & +1 \\ \mathcal{P}_-\psi_- & -1 & +1 \\ \overline{\mathcal{P}}_-\psi_- & +1 & -1 \end{array} \tag{1}$$

where

$$\mathcal{P}_\pm = \frac{1}{2}(1 - iI_\pm)$$

are the projectors $\mathcal{P}_\pm : T \otimes \mathbb{C} \to T_\pm^{(1,0)}$. The kinetic terms are

$$g(\psi_+, D_z \mathcal{P}_+ \psi_+) = g_{i\bar{j}} \psi_+^{\bar{j}} D_z \psi_+^i \,, \qquad g(\psi_-, D_{\bar{z}} \mathcal{P}_- \psi_-) = g_{i\bar{j}} \psi_-^{\bar{j}} D_{\bar{z}} \psi_-^i \,. \tag{2}$$

Note the slight abuse of notation in the right hand sides of the above equations as the decomposition in holomorphic and antiholomorphic components of $T \otimes \mathbb{C}$ is not unique. The table (1) and the kinetic terms (2) yield the fermion zero mode mismatches

$$\begin{aligned} \text{Vector (V)} &: \quad -\dim \mathrm{Ker} D_z^+ + \dim \mathrm{Ker} D_{\bar{z}}^+ - \dim \mathrm{Ker} D_{\bar{z}}^- + \dim \mathrm{Ker} D_z^- \,, \\ \text{Axial (A)} &: \quad -\dim \mathrm{Ker} D_z^+ + \dim \mathrm{Ker} D_{\bar{z}}^+ + \dim \mathrm{Ker} D_{\bar{z}}^- - \dim \mathrm{Ker} D_z^- \,. \end{aligned}$$

The superscript $+$ or $-$ keeps track of the tangent bundle to which the operators couple. Before performing the twist, these operators are Dirac operators, and we have the anomalies

$$(\text{Anomaly})_{\mathrm{V/A}} = \nu^+(D) \mp \nu^-(D) \,, \tag{3}$$

where $\nu^\pm(D)$ are the indeces of the Dirac operators coupled to the tangent bundles $T_\pm^{(1,0)}$. These indeces are given by the index theorem,

$$\nu^\pm(D) = \int_\Sigma \phi^* c_1(T_\pm^{(1,0)}) \,,$$

where ϕ is a map from a Riemann surface Σ to the target space X. Therefore, we find

$$(\text{Anomaly})_{\mathrm{V/A}} = \int_\Sigma \phi^* \big\{ c_1(T_+^{(1,0)}) \mp c_1(T_-^{(1,0)}) \big\} = \int_\Sigma \phi^* c_1(L_{2/1}) \,. \tag{4}$$

$L_{1/2}$ are the maximal isotropic subbundles of $(T \oplus T^*) \otimes \mathbb{C}$ coming from the generalized complex structures

$$\mathcal{J}^b_{1/2} = e^b \, \frac{1}{2} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix} e^{-b} \,, \qquad e^b = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \,,$$

which are given by

$$\begin{aligned} L_1 &= L_1^+ \oplus L_1^- = \left\{ v + (b+g)v : v \in T_+^{(1,0)} \right\} \oplus \left\{ v + (b-g)v : v \in T_-^{(1,0)} \right\} \,, \\ L_2 &= L_2^+ \oplus L_2^- = \left\{ v + (b+g)v : v \in T_+^{(1,0)} \right\} \oplus \left\{ v + (b-g)v : v \in T_-^{(0,1)} \right\} \,. \end{aligned}$$

The equations (4) have been found by Kapustin and Li [4]. They show the existence of a vector anomaly in addition to the axial anomaly, and imply that the geometry of the A/B model is constrained via $c_1(L_{2/1}) = 0$. When $I_+ = I_-$, one finds the known fact that the B model can only be defined on a Calabi-Yau manifold while no condition is required to define the A model.

The A and B models have vector and axial symmetries. These symmetry are spoiled at the quantum level by the presence of an anomaly, which is given in terms of the indeces of the Dolbeault operators. In fact, the equations (3) become

$$(\text{Anomaly})^{\text{tw}}_{\text{V/A}} = \nu^+(\bar\partial) \mp \nu^-(\bar\partial) \,.$$

The Dirac operator has become a Dolbeault operator as a result of the twist, and the superscript tw indicates that we are computing an anomaly for a twisted theory. The index theorem asserts that

$$\nu^\pm(\bar\partial) = \int_\Sigma \text{ch}(\phi^* T_\pm^{(1,0)}) \text{td}(T_\Sigma) = \dim_{\mathbb{C}} X (1-g) + \int_\Sigma \phi^* c_1(T_\pm^{(1,0)})$$

for a Riemann surface Σ of genus g. We have used that in the expansion of the Chern character, only forms of degree less than or equal to two enter since the bundles $\phi^* T_\pm^{(1,0)}$ are over Σ. Moreover, $\phi^* T_\pm^{(1,0)}$ have rank equal to $\dim_{\mathbb{C}} X$. The anomalies take the form

$$\begin{aligned} (\text{Anomaly})^{\text{tw}}_{\text{V}} &= \int_\Sigma \phi^* \left\{ c_1(T_+^{(1,0)}) - c_1(T_-^{(1,0)}) \right\} = \int_\Sigma \phi^* c_1(L_2) \,, \\ (\text{Anomaly})^{\text{tw}}_{\text{A}} &= 2\dim_{\mathbb{C}} X (1-g) + \int_\Sigma \phi^* \left\{ c_1(T_+^{(1,0)}) + c_1(T_-^{(1,0)}) \right\} \\ &= 2\dim_{\mathbb{C}} X (1-g) + \int_\Sigma \phi^* c_1(L_1) \,. \end{aligned}$$

Note that the vector anomaly is present only in the B model with $c_1(L_2) \neq 0$. Taking into account the vector and axial R-charges of the operators in the correlation functions, these anomalies lead to selection rules. The total anomaly is

$$
\begin{aligned}
(\text{Total anomaly})^{\text{tw}} &= (\text{Anomaly})^{\text{tw}}_{\text{V}} + (\text{Anomaly})^{\text{tw}}_{\text{A}} \\
&= 2\dim_{\mathbb{C}} X(1-g) + \int_\Sigma \phi^* \big\{ c_1(L_1) + c_1(L_2) \big\} \,.
\end{aligned}
$$

Imposing the geometry constraints, we get

$$
(\text{Total anomaly})^{\text{tw}}_{\text{A/B}} = 2\dim_{\mathbb{C}} X(1-g) + \int_\Sigma \phi^* c_1(L_{1/2}) \tag{5}
$$

for the A and B models respectively. We shall see that these equations are related to the dimension of the moduli spaces of generalized holomorphic maps.

3 The dimension of the moduli space

We want to compute the dimension of the moduli space of generalized holomorphic maps by using the GRR formula. First, recall that the instantons of the A and B models are given by [1, 4]

$$
\frac{1}{2}(1 - i\mathcal{J}^b_{2/1}) \begin{pmatrix} i\partial_2 \phi \\ g\partial_1 \phi \end{pmatrix} = 0 \,.
$$

This can be rewritten as

$$
\mathcal{J}^b_{1/2}(i \circ d\phi) = (i \circ d\phi) I_\Sigma \,,
$$

where I_Σ is the complex structure of the Riemann surface, and

$$
T_\Sigma \xrightarrow{d\phi} T_X \overset{i}{\hookrightarrow} T_X \oplus T^*_X \,.
$$

One can use the projection operators

$$
\mathcal{P}_{L_{1/2}} = \frac{1}{2}(1 - i\mathcal{J}_{1/2}) \,, \qquad \mathcal{P}_\Sigma = \frac{1}{2}(1 - i I_\Sigma) \,,
$$

to recast the above equation into

$$
\mathcal{P}_{L_{1/2}}(i \circ d\phi) = (i \circ d\phi)\mathcal{P}_\Sigma \,. \tag{6}
$$

Since $(T_X \oplus T_X^*) \otimes \mathbb{C} = L_{1/2} \oplus \overline{L}_{1/2}$, we have

$$(i \circ d\phi) = (i \circ d\phi)_{L_{1/2}} + (i \circ d\phi)_{\overline{L}_{1/2}} \,,$$

with $(i \circ d\phi)_{L_{1/2}}$ and $(i \circ d\phi)_{\overline{L}_{1/2}}$ sections of $L_{1/2}$ and $\overline{L}_{1/2}$. Substituting the last equation into (6), we get

$$(i \circ \bar{\partial}\phi)_{L_{1/2}} = (i \circ \partial\phi)_{\overline{L}_{1/2}} \,.$$

These are sections of orthogonal bundles. The only possibility is

$$(i \circ \bar{\partial}\phi)_{L_{1/2}} = 0 = (i \circ \partial\phi)_{\overline{L}_{1/2}} \,.$$

This indicates that infinitesimal deformations of generalized holomorphic maps are given in terms of $\bar{\partial}$-closed section of the bundle $\phi^* L_{1/2}$, *i.e.* they are elements of

$$H^{0,0}_{\bar{\partial}}(\Sigma, \phi^* L_{1/2}) = \check{H}^0(\Sigma, \phi^* L_{1/2}) \,.$$

Therefore, the dimension of the moduli space $\mathcal{M}_g^{\mathrm{A/B}}$ of generalized holomorphic maps from a Riemann surface of genus g to the target space of the A/B model is given by

$$\mathrm{dim}\mathcal{M}_g^{\mathrm{A/B}} = \mathrm{dim}\check{H}^0(\Sigma, \phi^* L_{1/2}) \,.$$

By the GRR formula,

$$\mathrm{dim}\check{H}^0(\Sigma, \phi^* L_{1/2}) - \mathrm{dim}\check{H}^1(\Sigma, \phi^* L_{1/2}) = \int_\Sigma \mathrm{ch}(\phi^* L_{1/2})\mathrm{td}(T_\Sigma) \,.$$

Considering that $(T_X \oplus T_X^*) \otimes \mathbb{C} = L_1 \oplus \overline{L}_2$, we get

$$\mathrm{dim}\mathcal{M}_g^{\mathrm{A/B}} = \mathrm{dim}\check{H}^1(\Sigma, \phi^* L_{1/2}) + 2\mathrm{dim}_{\mathbb{C}} X(1-g) + \int_\Sigma \phi^* c_1(L_{1/2}) \,.$$

If the moduli space of generalized holomorphic maps is unobstructed, *i.e.*

$$\check{H}^1(\Sigma, \phi^* L_{1/2}) = 0 \,,$$

we get

$$\mathrm{dim}\mathcal{M}_g^{\mathrm{A/B}} = 2\mathrm{dim}_{\mathbb{C}} X(1-g) + \int_\Sigma \phi^* c_1(L_{1/2}) \,.$$

Comparing this equation with (5), we have shown that in the unobstructed case the total anomaly coincides with the dimension of the moduli space of generalized holomorphic maps.

In this paper we will not investigate criteria under which the unobstructedness assumption holds. Instead, we simply assume from now on that under suitable conditions the moduli space of generalized holomorphic maps is unobstructed. Then, the equations above imply the triviality at $g > 1$ of the partition function of topological models defined on a generalized Calabi-Yau manifold, for which $c_1(L_1) = 0 = c_1(L_2)$. The partition function becomes non-trivial by coupling to worldsheet gravity. Consequently, we also have to integrate over the moduli space $\overline{\mathcal{M}}_{g,n}$ of curves with n marked points. This enhances the dimension of the moduli space, yielding the so-called virtual dimension. We indicate by $\overline{\mathcal{M}}_{g,n}^{\mathrm{A/B}}$ the moduli space of generalized holomorphic maps from Riemann surfaces with n marked points to a generalized Kähler manifold with $c_1(L_{2/1}) = 0$. From a physical viewpoint, n is the number of operator insertions. In other words, an n-point correlation function of topological string theory is obtained by integrating over $\overline{\mathcal{M}}_{g,n}^{\mathrm{A/B}}$. Since

$$\dim_{\mathbb{C}}\mathcal{M}_{g,n} = 3(g-1) + n\,,$$

the virtual dimension is

$$\dim\overline{\mathcal{M}}_{g,n}^{\mathrm{A/B}} = 2(\dim_{\mathbb{C}}X - 3)(1-g) + 2n + \int_{\Sigma} \phi^* c_1(L_{1/2})\,.$$

For generalized Calabi-Yau threefolds of complex dimension three

$$\dim\overline{\mathcal{M}}_{g,0}^{\mathrm{A/B}} = 0\,.$$

This suggests that the computation of the partition function reduces to a counting problem. If we consider the topological A model on a generalized Calabi-Yau manifold and take $I_+ = I_-$, we get the A model on a Calabi-Yau manifold, where the counting problem is know to lead to the Gromow-Witten invariants. They are symplectic invariants of the underlying Calabi-Yau manifold. It is conceivable that the counting problem in the generalized case might lead to new invariants. The invariants of the A model on a generalized Calabi-Yau manifold will degenerate to the Gromow-Witten invariants in the extreme case of $I_+ = I_-$. But in the case $I_+ \neq I_-$ such invariants will be of generalized type, which interpolates between those of symplectic and complex type.

4 Conclusions and outlook

An important issue that deserves to be studied is to find the criteria under which the moduli space is unobstructed. In fact, we have seen that in such a case generalized Calabi-Yau threefolds assume a special role. One should note that for holomorphic maps into a Calabi-Yau threefold, there are often obstructions that yield a positive virtual dimension.

In the unobstructed case of holomorphic maps from curves with no marked points to a Calabi-Yau threefold, the virtual dimension is zero. Therefore, the moduli space is given by a number of points, and the computation of the partition function reduces to a counting point problem that is known to lead to Gromov-Witten invariants. We have seen above that in the Hitchin's generalized setup under suitable conditions the virtual dimension is zero. This indicates that also in this case the computation of the partition function reduces to a counting point problem that could lead to a generalization of Gromov-Witten theory. This idea is supported by the physics fact that the A model on a generalized Calabi-Yau threefold with the two complex structures identified is nothing but the A model on a Calabi-Yau threefold. Therefore, the generalized Gromov-Witten theory, which might exist when the two complex structures are different, has to degenerate to the usual one when the two structures are equal.

The computation of the partition function is highly non trivial. Therefore, one should start with genus zero correlators. For usual genus zero holomorphic maps, selection rules assert that only the three-point correlator of three $(1,1)$-forms is non zero. This correlator leads to the deformed intersection theory. In other words, the intersection number of three divisors of a Calabi-Yau threefold is deformed by the quantum cohomology ring of the target space, which contains genus zero Gromov-Witten invariants. In the generalized setup the situation should be quite similar. However, the $(1,1)$-forms should be replaced by Hitchin's polyforms of the target space, and one should ask first what the intersection number of polycycles could be. The next step would consist in seeing how the quantum cohomology ring deforms this generalized intersection number.

5 Acknowledgments

The author would like to thank Albrecht Klemm and Stefan Theisen for interesting discussions, and Florian Gmeiner and Frederik Witt for proofreading the paper. This work was supported by SFB 647.

References

[1] A. Kapustin, *Topological strings on noncommutative manifolds*, *Int. J. Geom. Meth. Mod. Phys.* **1** (2004) 49, [hep-th/0310057].

[2] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Q. J. Math.* **54 no.3** (2003) 281, [math.DG/0209099].

[3] Gualtieri, *Generalized complex geometry.* Oxford University DPhil thesis, 2004, [math.DG/0401221].

[4] A. Kapustin and Y. Li, *Topological sigma-models with H-flux and twisted generalized complex manifolds*, hep-th/0407249.